Università di Genova

ItaCa

Comonads for dependent types

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Dependent types

We want to be able to express types that vary with/depend on terms.

$\mathrm{Vect}_K[n]$ the type of K-vectors of length $n : N$

$$\frac{\vdash n : N \qquad N \vdash Vect_K\ Type}{\vdash Vect_K[n]\ Type}$$

$$(\mathrm{DTy})\ \frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B\ Type}{\Gamma \vdash B[a]\ Type}$$

First attempt: $B \to A$ morphism in a (lcc) category.

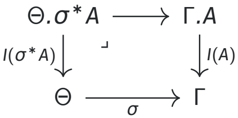
coherence issues, substitution/pullback must be strictly associative

Sophisticated solution: categories with structure.

Two models

CE-systems[1] (unstratified **C-systems**[2])

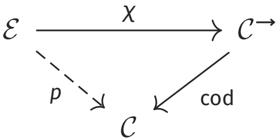
- two strict category structures $\mathcal{F}, \mathcal{C}$ with $Ob(\mathcal{F}) = Ob(\mathcal{C})$
- id-on-obj $I : \mathcal{F} \to \mathcal{C}$
- a chosen 1 terminal in $\mathcal{F}$
- for any $\sigma : \Theta \to \Gamma$ in $\mathcal{C}$ and $A \in \mathcal{F}_{/\Gamma}$ a functorial choice of a pullback



plus equations.

Comprehension categories[3]

- a category $\mathcal{C}$
- a functor $\chi : \mathcal{E} \to \mathcal{C}^{\to}$ s.t. $\mathrm{cod} \circ \chi$ is a Grothendieck fibration and χ sends cartesian maps to pullback squares such that $\sigma^* A \in \mathcal{F}_{/\Theta}$



[1] Ahrens et al., "B-systems and C-systems are equivalent", 2021.
[2] Voevodsky, "Subsystems and regular quotients of C-systems", 2014.
[3] Jacobs, "Comprehension categories and the semantics of type dependency", 1993.

Two models

CE-systems		**Comprehension categories**
$\mathcal{C}, \mathcal{F}, I, 1$, functorial choice of pb		$\mathcal{C}, \chi$
Γ in $\mathcal{C}$	$\vdash \Gamma\ Ctx$	Γ in $\mathcal{C}$
A in $\mathcal{F}$ and $\mathrm{cod} I(A) = \Gamma$	$\Gamma \vdash A\ Type$	A in $\mathcal{E}$ s.t. $p(A) = \Gamma$
$\mathrm{dom} I(A)$	$\Gamma.A$	$\mathrm{dom} \circ \chi_A$
sections of $I(A)$	$\Gamma \vdash a : A$	sections of χ_A

Examples

$$\chi : \mathcal{E} \to \mathcal{C}^{\to}$$

Term model: $\mathcal{C}$ the category of α-equivalence classes of contexts and terms, $\mathcal{E}$ typed judgements and substitutions, $\chi : (\Gamma \vdash A) \mapsto$ (the projection of $\Gamma.A$ on Γ)

Display-categories: $\mathcal{C}$ a category, D a collection of morphisms in $\mathcal{C}$ such that (it has and) it is closed for pullback along any map in $\mathcal{C}$, $\chi : D \hookrightarrow \mathcal{C}$. If D is monos, we call $D = Sub(\mathcal{C})$.

Simple fibration: $\mathcal{C}$ with products, $s(\mathcal{C})$ the simple category on $\mathcal{C}$, we can define $\chi : (I, X) \mapsto (\pi_1 : I \times X \to I)$.

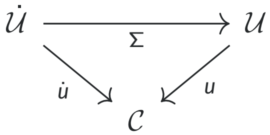
Topos comprehension: $\mathcal{C}$ a topos, Ω its sub-object classifier, then we can define $\{-\} : \mathcal{C}_{/\Omega} \to \mathcal{C}^{\to}$ obtained by pullback along $t : 1 \to \Omega$.

A shift in perspective

What if we consider terms as separate objects?
Categories with families[4], **natural models**[5]: types are sets indexed over contexts, terms are sets indexed over types.

$$\text{(DTy)}\ \frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B}{\Gamma \vdash B[a]} \qquad\qquad \text{(DTy)}\ \frac{\Gamma \vdash a : A\ \mathit{Term} \qquad \Gamma.A \vdash B\ \mathit{Type}}{\Gamma \vdash B[a]\ \mathit{Type}}$$

"Encode $\vdash$-relation in a discrete fibration, then rules are commutative triangles."



- u collects types: write $\Gamma \vdash A\ \mathit{Type}$ *for* $u(A) = \Gamma$
- $\dot{u}$ collects terms: write $\Gamma \vdash a : A$ *for* $\dot{u}(a) = \Gamma$ and $\Sigma(a) = A$

such a triangle encodes $\dfrac{\Gamma \vdash a : \Sigma(a)}{\Gamma \vdash \Sigma(a)\ \mathit{Type}}$

[4]Dybjer, "Internal type theory", 1996.
[5]Awodey, "Natural models of homotopy type theory", 2018.

A shift in the shift in perspective

$$(\mathrm{DTy})\ \frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B}{\Gamma \vdash B[a]} \qquad\qquad (\mathrm{DTy})\ \frac{\Gamma \vdash a : A\ \mathit{Term} \qquad \Gamma.A \vdash B\ \mathit{Type}}{\Gamma \vdash B[a]\ \mathit{Type}}$$

"Encode ⊢-relation in a discrete fibration, then rules are commutative triangles."

1. Why discrete?
   Many of the examples we have aren't, still a good interpretation.
2. If a rule is a diagram, is each diagram a rule?
3. What does it mean to manipulate rules?
4. What deductive power do we need?

Judgemental theories

“Encode ⊢-relation in a *(possibly non discrete)* fibration,
then rules are *lax* commutative triangles.”

Following this motto, we define **judgemental theories**[6]:

1. define basic ⊢-encoding fibrations (e.g. u, $\dot{u}$);
2. define basic rules relating them (e.g. Σ);
3. encode deductive power into the system using categorical constructions (e.g. below).

$$\begin{array}{ccccc} \mathrm{Eq}(pr_1, pr_2) & \longrightarrow & \dot{\mathcal{U}} \times \dot{\mathcal{U}} & \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} & \dot{\mathcal{U}} \\ \uparrow & \nearrow_{diag} & & & \\ \dot{\mathcal{U}} & & & & \end{array}$$

reads as

$$(\rho)\ \frac{\Gamma \vdash a : A}{\Gamma \vdash a =_A a}$$

[6] Coraglia e Di Liberti, “Context, Judgement, Deduction”, 2022*.

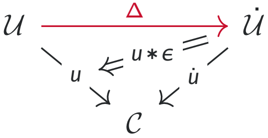
Judgemental dtts

Definition: the judgemental theory of dependent types

A *pre-judgemental dtt* is the data of fibrations $u, \dot{u}$, a morphism of fibrations $\Sigma : \dot{u} \to u$, and Δ right adjoint to Σ with cartesian unit and counit.
A *judgemental dtt* is the smallest 2-subcategory of **Cat** containing $u, \dot{u}, \Sigma, \Delta, \eta, \epsilon$ and being closed under finite limits and $\sharp$-lifting.

Theorem (C. - Di Liberti)

A judgemental dtt contains codes for all structural rules of dependent type theory.



reads as

$$(\Delta)\ \frac{\Gamma \vdash A}{\Gamma.A \vdash x_A : A(u * \epsilon)_A}$$

Coding dependent families

$$\Gamma \vdash a : A \quad \Gamma.A \vdash b : B$$

$$\Gamma \vdash A \quad \Gamma.A \vdash b : B$$

$$\Gamma \vdash a : A \quad \Gamma.A \vdash B$$

$$\Gamma \vdash A \quad \Gamma.A \vdash B$$

$$\begin{array}{ccccccc}
\dot{\mathcal{U}}.\Sigma\Delta\dot{\mathcal{U}} & \longrightarrow & \mathcal{U}.\Delta\dot{\mathcal{U}} & & \longrightarrow & & \dot{\mathcal{U}} \\
\downarrow \ulcorner & & \downarrow \ulcorner & & & & \downarrow \Sigma \\
\dot{\mathcal{U}}.\Sigma\Delta\mathcal{U} & \longrightarrow & \mathcal{U}.\Delta\mathcal{U} & & \longrightarrow & & \mathcal{U} \\
\downarrow \ulcorner & & \downarrow \ulcorner & & & & \downarrow u \\
\dot{\mathcal{U}} & \xrightarrow[\Sigma]{} & \mathcal{U} & \xrightarrow[\Delta]{} & \dot{\mathcal{U}} & \xrightarrow[\dot{u}]{} & \mathcal{C}
\end{array}$$

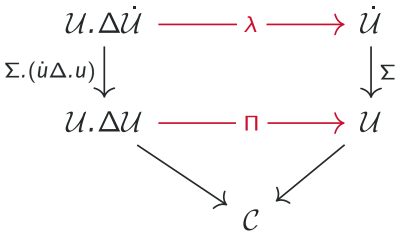
$$a : A \longmapsto A \longmapsto x_A : A^+ \longmapsto \Gamma.A$$

Type constructors

Plus, we can define what diagrams one needs to add to jDTT in order to get type constructors.

Theorem

It has Π-*types* if it has two additional rules Π, λ such that the diagram below is commutative and the upper square is a pullback.



$$(\Pi I)\ \frac{\Gamma \vdash A \qquad \Gamma.A \vdash b : B}{\Gamma \vdash \lambda_A b : \Pi_A B}$$

$$(\Pi F)\ \frac{\Gamma \vdash A \qquad \Gamma.A \vdash B}{\Gamma \vdash \Pi_A B}$$

(ΠE): the unique map induced by the pullback from the classifier of (A, B, f, a)
(ΠCη) and (ΠCβ): induced by the canonical isomorphism

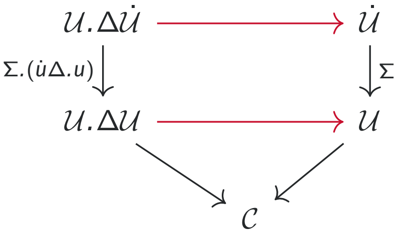
Recipe for type constructors

This generalizes quite nicely.

Pick

$$\Gamma \vdash A \qquad \Gamma.A \vdash b : B$$

$$\Gamma \vdash A \qquad \Gamma.A \vdash B$$



... get Π-types.

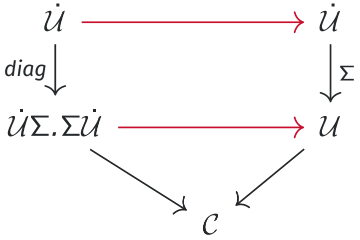
Recipe for type constructors

This generalizes quite nicely.

Pick

$$\Gamma \vdash a : A$$

$$\Gamma \vdash a : A \quad \Gamma \vdash a' : A$$



... get Id-types.

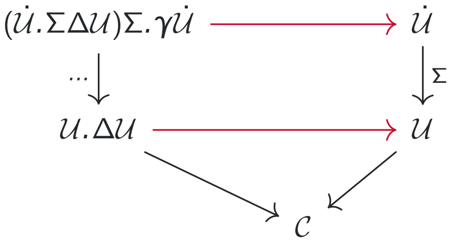
Recipe for type constructors

This generalizes quite nicely.

Pick

$$\Gamma \vdash A \quad \Gamma.A \vdash B \quad \Gamma \vdash a : A \quad \Gamma \vdash b : B[a]$$

$$\Gamma \vdash A \quad \Gamma.A \vdash B$$



... get sum-types.

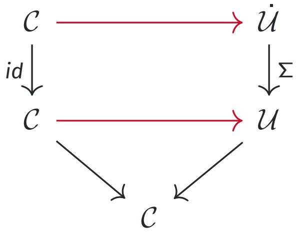
Recipe for type constructors

This generalizes quite nicely.

Pick

$$(\vdash \Gamma \; ctx)$$

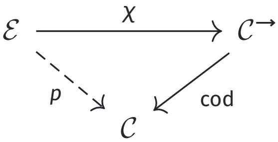
$$(\vdash \Gamma \; ctx)$$



... get unit-types.

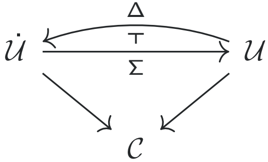
Where were we?

Comprehension categories



few assumptions, elegant
computations are “internal”
how to add (non trivial) constructors?

jDTTs



more structure
can read computations in **Cat**
can easily encode constructors

Theorem (C. - Emmenegger)

There is a 2-equivalence **CompCat** $\equiv$ **jDTT**.

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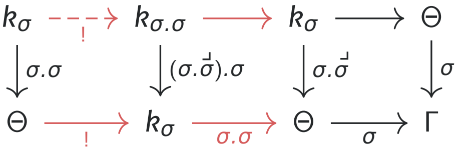
Sections are coalgebras

Let $\mathcal{D}$ a category with pullbacks.
Then we can define a comonad of kernel-pairs

$$K : \mathcal{D}^{\to} \to \mathcal{D}^{\to},\ \sigma \mapsto \sigma.\sigma$$

ϵ_σ : right square

δ_σ : left square



using the UP twice

and coalgebras with carrier σ are precisely sections of σ.

Weakening and contraction comonads

Comprehension categories	**jDTTs**
(kind of) the kernel-pair comonad	$\Sigma\Delta$

Definition: wc-comonad[7]

Let $p : \mathcal{E} \to \mathcal{C}$ a fibration. A *weakening and contraction comonad on p* is a comonad (K, ϵ, δ) on $\mathcal{E}$ such that

1. the components of ϵ are p-cartesian and
2. for every cartesian arrow $f : A \to B$ in $\mathcal{E}$ the image in $\mathcal{C}$ under p of the naturality square is a pullback square.

$$\begin{array}{ccc} pKA & \xrightarrow{p\epsilon_A} & pA \\ {\scriptstyle pKf}\downarrow & & \downarrow{\scriptstyle pf} \\ pKB & \xrightarrow{p\epsilon_B} & pB \end{array}$$

[7] Jacobs, Categorical logic and type theory, 1999.

Weakening and contraction comonads

Definition: wc-comonad

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$$
\begin{array}{ccc}
pKA & \xrightarrow{p\epsilon_A} & pA \\
{\scriptstyle pKf}\downarrow & & \downarrow{\scriptstyle pf} \\
pKB & \xrightarrow{p\epsilon_B} & pB
\end{array}
$$

Actually δ is canonically determined by (K, ϵ): the naturality square of ϵ at ϵ_A

- is over a pullback in $\mathcal{C}$
- has two parallel cartesian sides

hence it is itself a pullback.

Weakening and contraction comonads

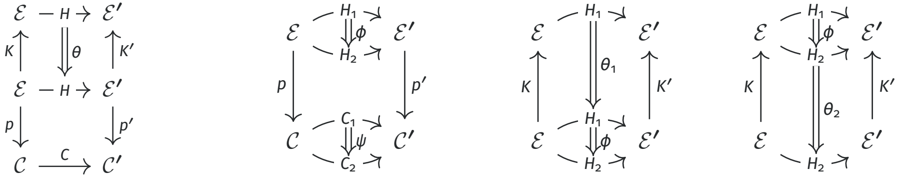
weakening	$\epsilon_A : KA \longrightarrow A$	add a “dummy” variable to pA
contraction	$\delta_A : KA \longrightarrow KKA$	if $x : A, y : A$ we can collapse them to one of the two (i.e. substitute $[y/x]$)
	comonad equations	weakening then contracting does nothing (up to α-equivalence)

The 2-category **wcCmd**

A *0-cell* is a pair $\mathbb{K} = (p, K)$.
A *1-cell* $\mathbb{K} \to \mathbb{K}'$ is a triple (H, C, θ) as in the diagram below, such that

1. $(H, C) \colon p \to p'$ is a 1-cell in **Fib**
2. (H, θ) is a lax morphism of comonads.

A *2-cell* between $(H_1, C_1, \theta_1) \Rightarrow (H_2, C_2, \theta_2)$ is a 2-cell $(\phi, \psi) \colon (H_1, C_1) \Rightarrow (H_2, C_2)$ in **Fib** as in the diagram below, such that $\phi * \theta_1 = \theta_2 * \phi$.



We write $\mathbf{wcCmd_{ps}}$ (resp. $\mathbf{wcCmd_{str}}$) for the 2-full 2-subcategories of **wcCmd** with the same 0-cells, and only those 1-cells (H, C, θ) such that (H, θ) is a pseudo (resp. strict) morphism of comonads.

wcCmd ≡ CompCat

Lemma 1 (Jacobs)

Each wc-comonad induces a comprehension category and viceversa.

- For $\chi : \mathcal{E} \to \mathcal{C}^{\to}$, $p = cod \circ \chi$,
  define K on p as: $KE := (\chi_E)^* E$, $\epsilon_E := \overline{\chi_E}$.
- For $(p : \mathcal{E} \to \mathcal{C}, K)$,
  define $\chi : \mathcal{E} \to \mathcal{C}^{\to}$ as $\chi(E) := p\epsilon_E$.

$$\begin{array}{ccc} (\chi_E)^* E & \xrightarrow{\overline{\chi_E}} & E \\ & & \\ \bullet & \xrightarrow{\chi_E} & pE \end{array} \qquad \begin{array}{c} \mathcal{E} \\ \downarrow p \\ \mathcal{C} \end{array}$$

Theorem 2 (C. - Emmenegger)

Lemma 1 upgrades to a 2-equivalence[8] **wcCmd ≡ CompCat**, which restricts to the $-_{\mathbf{ps}}$ (resp. $-_{\mathbf{str}}$) subcategories.

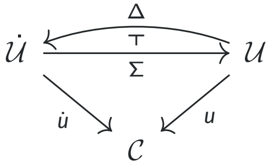
[8]With a suitable choice of cells for **CompCat**. In the literature, **CompCat** is used for $\mathbf{CompCat}_{\mathbf{str}}$.

wcCmd ≡ jDTT

Lemma 3 (C. - Emmenegger)

Each wc-comonad induces a jDTT and viceversa.

- For $u, \dot{u}, \Sigma, \Delta$ jDTT, the comonad



$$(\Sigma\Delta : \mathcal{U} \to \mathcal{U}, \epsilon)$$

  is wc on u.
- For $(p : \mathcal{E} \to \mathcal{C}, K)$, we can compute the Eilenberg-Moore adjunction

$$CoAlg(K) \underset{U}{\overset{C}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathcal{E}$$

  which extends to a jDTT with fibrations p and $p \circ U$.

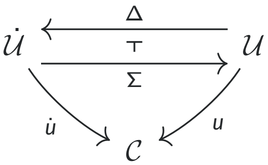
Key: If $e : E \to KE$ is a coalgebra for K wc-comonad on p, then e is p-cartesian.

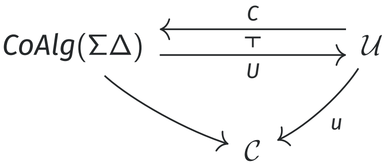
wcCmd $\equiv$ **jDTT**

[Wannabe] Theorem

Lemma 3 upgrades to a 2-equivalence[9] **wcCmd** $\equiv$ **jDTT**, which restricts to the $-_{\mathbf{ps}}$ (resp. $-_{\mathbf{str}}$) subcategories.

Starting from a wc-comonad, one construction after the other yields the identity on-the-nose.For the opposite composition we need to compare:





[9] With an appropriate choice of cells for **jDTT** induced by the established comonad morphisms.

Every term is a coalgebra, every coalgebra is a term

- Each a in $\dot{\mathcal{U}}$ produces a coalgebra on $\Sigma a = A$, $\Sigma\eta_a : A \to \Sigma\Delta A$.
- What about the converse? To each $h : A \to \Sigma\Delta A$ we want to match an a_h in $\dot{\mathcal{U}}$.

$$a_h := \text{the domain of the } \dot{u}\text{-cartesian lift of } u(h) \text{ at } \Delta A$$

$$\begin{array}{ccc} a_h & \xrightarrow{\overline{u(h)}} & \Delta A \\ \\ A & \xrightarrow{h} & \Sigma\Delta A \\ \\ \Gamma & \xrightarrow{u(h)} & \Gamma.A \end{array} \qquad \begin{array}{c} \dot{\mathcal{U}} \\ \Sigma \downarrow \quad \searrow \dot{u} \\ \mathcal{U} \\ u \downarrow \\ \mathcal{C} \end{array}$$

Notice that $\overline{u(h)}$ is cartesian by hypothesis, and Σ preserves cartesian maps, hence we have both h and $\Sigma(\overline{u(h)})$ cartesian and over $u(h)$. They are isomorphic in $\mathcal{U}_{/\Sigma\Delta A}$ by uniqueness of the cartesian lift.

By similar uniqueness arguments we can prove $a_{\Sigma\eta_a} \cong a$ and $h \cong \Sigma\overline{u(h)} \cong \Sigma\eta_{a_h}$.

Lemma 4

$\dot{\mathcal{U}}$ and $CoAlg(\Sigma\Delta)$ are equivalent categories.

CompCat $\equiv$ jDTT

Theorem 5 (C. - Emmenegger)

Lemma 3 induces to a 2-equivalence **wcCmd** $\equiv$ **jDTT** which restricts to the $-_{\mathbf{ps}}$ (resp. $-_{\mathbf{str}}$) subcategories.

Corollary

There is a 2-equivalence **CompCat** $\equiv$ **jDTT** which restricts to the $-_{\mathbf{ps}}$ (resp. $-_{\mathbf{str}}$) subcategories.

The moment we choose how to interpret weakening/context extension (χ in a comprehension category), we have immediately bounded:

- how we shall interpret contraction (wc-comultiplication);
- how we shall interpret terms (terms in a jDTT);
- how we shall interpret type constructors (constructing types in a jDTT).

Back to CE-systems: sums?

… and $A \in \mathcal{F}_{/\Gamma}$ a **functorial** *choice of a pullback square such that $\sigma^* A \in \mathcal{F}_{/\Theta}$.*

$$\Gamma.A.B \xrightarrow{I(B)} \Gamma.A \xrightarrow{I(A)} \Gamma, \qquad \Gamma.A.B \xrightarrow{I(A \circ B)} \Gamma$$

CE-systems seem to have *structurally encoded* into them some sort of dependent sum construct. We can now make this statement precise.

1. Relate CE-systems to jDTTs.
2. Show that such jDTTs have dependent sums.

CE-systems are jDTTs

Theorem 6 (C. - Emmenegger)

There is an adjunction $L : \mathbf{jDTT_{str}} \leftrightarrows \mathbf{CEsys} : R$.

We are presently interested in R. To a CE-system $I : \mathcal{F} \to \mathcal{C}$ we map the following:

- the fibration $p = \mathrm{cod} \circ I^{\to} : \mathcal{F}^{\to} \to \mathcal{C}$;
- the wc-comonad on p induced by the kernel-pair construction:

$$K_F : A \mapsto I(A)^* A.$$

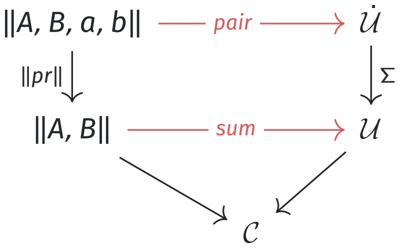
By hypothesis on I, $I(A)^* A \in \mathcal{F}^{\to}$.

CE-systems have dependent sums as jDTTs, I

Recat that a jDTT has dependent sums iff there are functors *sum*, *pair* s.t. the diagram here commutes and the top square is a pullback.

$$(\mathit{sum}\mathrm{I})\ \frac{\Gamma \vdash A \qquad \Gamma.A \vdash B \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : B[a]}{\Gamma \vdash \mathit{pair}(a, b) : \mathit{sum}_A B}$$

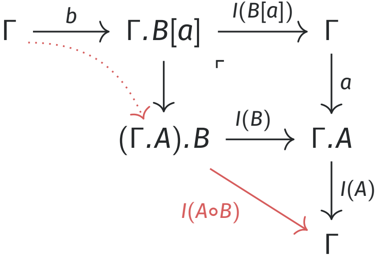
$$(\mathit{sum}\mathrm{F})\ \frac{\Gamma \vdash A \qquad \Gamma.A \vdash B}{\Gamma \vdash \mathit{sum}_A B}$$



Then we compute both categories via iterated pullbacks and show that we can define such *sum*, *pair*.

CE-systems have dependent sums as jDTTs, II

Objects of $\|A, B, a, b\|$ are diagrams as this



for $A, B, B[a]$ in $\mathcal{F}$ and a, b sections in $\mathcal{C}$.The category $\|A, B\|$ only keeps track of the lower part of the diagram, and the desired vertical functor “forgets” the upper part.
Defining *sum*, *pair* as the red maps yields the desired functors.

CE-systems have dependent sums as jDTTs, III

Not only that,

Proposition

A jDTT with dependent sums is equivalent through $L : \mathbf{jDTT_{str}} \leftrightarrows \mathbf{CEsys} : R$ to a CE-system.

so that it is a characterizing property.

In summation

Through a **deeply syntactic approach** we have:

- described a new model that allows for easier treatment of type constructors;
- established its relation with previous models;
- provided a method for characterizing other models;
- described how comonads come into play;
- and that there is no escaping sections.

Still, there are plenty of things that should be looked into, for example:

- use the richer structure to study sub-typing (j/w F. Dagnino);
- extend the theory and the definition to type constructors not included (inductive, coinductive);
- prove *some* completeness result for jDTTs as a calculus;
- ...what about monads (e.g. $\Delta\Sigma$)?

Thank you for listening!